Grade 7/8 Math Circles
February 20/21/22/23, 2023
Math in Musical Scales

Applying Math to Music

When sitting in your math class, it may be hard to think about how the math you're learning can apply to the real world. From banking to engineering to aviation, math has many different applications and is the basis for almost everything. Today, we're going to focus on an application that interests many different people: music!

Review of Fractions

A fraction represents a certain number of parts out of a whole. We call the top part of the fraction the **numerator**. The numerator represents how many parts we have. The bottom part of the fraction is called the **denominator** and it represents how many parts are in a whole.

Fraction Multiplication

Multiplication is arguably the easiest operation to do on fractions. If we are multiplying two fractions together, we simply multiply their numerators together and multiply their denominators together. The resulting fraction's numerator will be the product of the two numerators and the denominator will be the product of the denominators.

Example A

To calculate $\frac{3}{4} \times \frac{3}{8}$, we compute:

$\frac{3}{4} \times \frac{3}{8} = \frac{3\times 3}{4\times 8} = \frac{9}{32}$

Thus $\frac{3}{4} \times \frac{3}{8}$ is $\frac{9}{32}$

Reciprocal

The **reciprocal** of a fraction is when the numerator becomes the denominator and the denominator becomes the numerator.

Example B

a) The reciprocal of $\frac{5}{8}$ is $\frac{8}{5}$.

b) The reciprocal of $\frac{1}{3}$ is $\frac{3}{1} = 3$.

c) The reciprocal of $7 = \frac{7}{1}$ is $\frac{1}{7}$.

Dividing Fractions

To divide fractions, we keep the first fraction the same but take the reciprocal of the second fraction. From there, we turn the division symbol into multiplication and perform fraction multiplication.

Example C

To calculate $\frac{7}{8} \div \frac{2}{3}$, we compute:

$\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \times \frac{3}{2} = \frac{7\times3}{8\times2} = \frac{21}{16}$

Thus $\frac{7}{8} \div \frac{2}{3}$ is $\frac{21}{16}$

Simplifying Fractions

After performing any operations on fractions, we always want to make sure that the final answer is in **simplest form** (or reduced form). This means that there are no common factors between the numerator and denominator besides 1.

Exercise 1

Determine whether or not the following fractions are in simplest form. If they aren't, write them in simplest form.

a) $\frac{2}{3}$ b) $\frac{3}{9}$ c) $\frac{64}{68}$ d) $\frac{5}{10}$ e) $\frac{36}{63}$ f) $\frac{7}{8}$ g) $\frac{12}{36}$ h) $\frac{57}{91}$

Intro to Scales

What is a scale?

A **scale** is a collection of musical notes arranged in increasing order. For the purposes of this lesson, each scale has 8 notes. This lesson will show us how to find the notes of a certain scale.

Terminology

There are a few key terms we need to know before we can learn more about scales:

- **Frequency**: How many times the sound wave of a note cycles in a given unit of time (also called **pitch**). In this lesson we will use Hertz (Hz) as our unit of frequency.
- **Tonic**: The starting note of a scale.
- **Octave**: Describes the series of the 8 notes in a scale. Two notes are an octave apart if the frequency of one note is double the frequency of the other note. Thus the first and last note in a scale are the same note, just an octave a part.
- **Semitone**: A single jump from one note to another (also called a **half step**). There are a total of 12 semitones in an octave.
- **Tone**: Two jumps from one note to another (also called a **whole step**). One tone is equal to two semitones.

Stop and Think

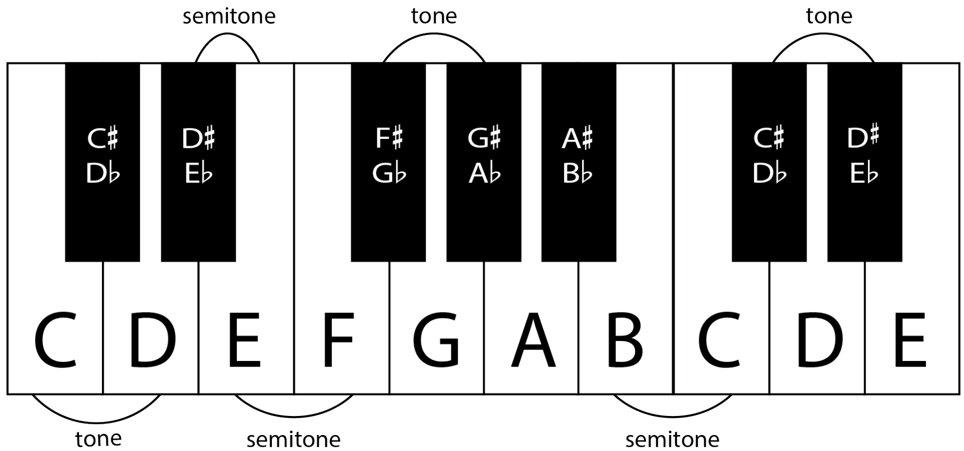
How are there 12 semitones in a scale when there are only 8 notes?

Take a look at the table below to see the general naming system for notes of any scale.

1^{st} **Note**	2^{nd} **Note**	3^{rd} **Note**	4^{th} **Note**	5^{th} **Note**	6^{th} **Note**	7^{th} **Note**	8^{th} **Note**
do	re	mi	fa	so	la	ti	do

Note Names

The note naming system in music uses the letters from A to G. Each note from A to G can also be sharpened of flattened. **Sharpening** the note makes the frequency higher by one semitone and **flattening** a note makes the frequency lower by one semitone. Sharp notes are donoted with $\sharp$ and flat notes are denoted with $\flat$. The black notes on a piano represent the flat and sharp versions of notes. Take a look at the piano below to see how whole steps and half steps relate to sharpening and flattening.



In general, two natural notes (notes that are not sharpened or flattened) are a tone apart with the exception of E and F and B and C which are semitones apart. Notice how all the black keys have two different names.

Stop and Think

Why is there no F flat on the diagram above?

How are scales formed?

Every scale starts with its tonic which is the building block for the rest of the scale. The notes in the scale are determined by the frequency of the tonic being multiplied by certain fractions. The fractions that we use are determined by which **tuning system** we use. Let's explore this more.

Tuning Systems

Pythagorean Tuning

One of the very first tuning systems that has been documented is **Pythagorean tuning**. This tuning system is based around the fraction $\frac{3}{2}$.

do	re	mi	fa	so	la	ti	do
1	$\frac{3^2}{2^3}$	$\frac{3^4}{2^6}$	$\frac{2^2}{3}$	$\frac{3}{2}$	$\frac{3^3}{2^4}$	$\frac{3^5}{2^7}$	2
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2

Example D

If we are given that the tonic note of a Pythagorean scale is 440 Hz, find the frequency of the fifth note.

The fraction corresponding to the fifth note or "so" is $\frac{3}{2}$. Thus, to find the frequency of that note we multiply 440 Hz $\times\frac{3}{2} = \frac{440\times 3}{2} = \frac{1320}{2} = 660$ Hz.

Exercise 2

Given that the frequency of the tonic is 300 Hz, find the rest of the notes in the Pythagorean scale.

Just Intonation

One problem with Pythagorean tuning is that some of the numbers in the fractions, like $\frac{243}{128}$, are quite large to work with. For this reason, people wanted to find a new tuning system that worked with smaller numbers. And so, the just intonation system was born.

do	re	mi	fa	so	la	ti	do
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

We notice that the second, fourth, and fifth note fractions in the just intonation system are the same as the Pythagorean tuning system. The difference here is that the bigger numerators and denominators in the third, sixth, and seventh notes are changed to smaller numbers.

Let's compare mi, la, and ti in the Pythagorean and just scales to see how closely they relate.

	mi	la	ti
Pythagorean scale	$\frac{81}{64} \approx 1.2656$	$\frac{27}{16} = 1.6875$	$\frac{243}{128} \approx 1.8984$
Just scale	$\frac{5}{4} = 1.25$	$\frac{5}{3} \approx 1.6667$	$\frac{15}{8} = 1.875$
Difference	0.015625	≈ 0.02083	0.0234375

Stop and Think

Do you think that small of a difference in fractions will result in an audible difference between the two tuning systems?

Example E

If we are given that the tonic note of a just intonation scale is 400 Hz, find the frequency of the seventh note.

The fraction corresponding to the seventh note or "ti" is $\frac{15}{8}$. Thus, to find the frequency of that note we multiply 400 Hz $\times \frac{15}{8} = \frac{400 \times 15}{8} = \frac{6000}{8} = 750$ Hz.

Exercise 3

Given that the frequency of the tonic is 280 Hz, find the rest of the notes in the just intonation scale.

Transposition

Every song is written in a certain **key**. This means that a song uses only the notes from a certain scale. The key is named after the tonic of the scale. So, for example, the key of C uses the scale starting at C.

Sometimes when playing a song, you may decide that you want to play it in a different key. This is called **transposing** a song. We can actually use the fractions in our tuning system to do this. To better understand this, let's walk through an example.

Example F

Transpose the given Pythagorean scale with a tonic of C to have a tonic of E. The frequency of C_1 is 261.6 Hz.

C_1	D	E	F	G	A	B	C_2
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2

Note that we use the **subscripts** on C to denote the different octaves.

We are only given the frequency of the tonic so the first thing we can do is calculate the frequencies of the rest of the notes for the key of C. To do this, we multiply 261.6 Hz by each of the fractions like we've done in previous examples. The resulting frequencies should be:

$\mathbf{C_1}$	**D**	**E**	**F**	**G**	**A**	**B**	$\mathbf{C_2}$
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2
261.6 Hz	294.3 Hz	331.1 Hz	348.8 Hz	392.4 Hz	441.5 Hz	496.7 Hz	523.2 Hz

Now, we want our scale to start on E. To do this, we treat E as our new tonic. Then, we would have to multiply each fraction by 331.1 Hz (which is the frequency of E we got from the table above). Our new transposed scale should look like this:

$\mathbf{E_1}$	**-**	**-**	**-**	**-**	**-**	**-**	$\mathbf{E_2}$
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2
331.1 Hz	372.5 Hz	419 Hz	441.5 Hz	496.7 Hz	558.7 Hz	628.6 Hz	662.2 Hz

We know that the new transposed scale starts and ends on E, but what about the rest of the notes? Can you see any frequencies from the original C scale that also appear in the new E scale? There are 2 frequencies from the C scale: A and B. And so our final scale will look like this:

$\mathbf{E_1}$	**-**	**-**	**A**	**B**	**-**	**-**	$\mathbf{E_2}$
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2
331.1 Hz	372.5 Hz	419 Hz	441.5 Hz	496.7 Hz	558.7 Hz	628.6 Hz	662.2 Hz

We might assume that the scale of E above would be E, F, G, A, B, C, D, E but we saw that the frequencies in the E scale besides E, A, and B didn't match the ones we found in the original C scale. So what should the rest of those notes be? Here is where sharpening and flattening notes come into play.

Stop and Think

Compare the frequency of the second note in the E scale to the frequency of F in the C scale. Do you think the unknown note in the E scale is flatter or sharper than F?

We can compare the frequencies of the unknown notes with the notes that we do know in the C scale. If the unknown note is a bit higher than the known note, then it is sharp. If the unknown note is a bit lower, then it is flat. Note that some of the unknown frequencies will be an octave higher than the known frequencies.

Exercise 4

Find the missing notes in the E scale from Example F.

Intervals

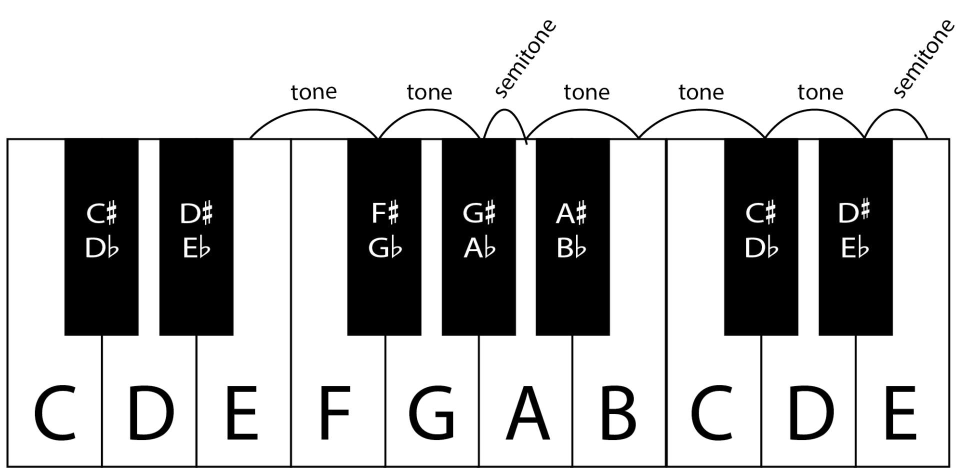
An alternate way to find these missing notes is by using intervals. The **intervals** of the scale describes the quotient between the fractions of two notes. It also describes whether a tone or semitone separates two notes in the scale. The intervals for the Pythagorean scale are as follows:

$$\left|\ \frac{9}{8}\ \right|\ \frac{9}{8}\ \left|\ \frac{256}{243}\ \right|\ \frac{9}{8}\ \left|\ \frac{9}{8}\ \right|\ \frac{9}{8}\ \left|\ \frac{256}{243}\ \right|$$

We notice that $\frac{9}{8} = 1.125$ and $\frac{256}{243} \approx 1.0535$ and so $\frac{9}{8} > \frac{256}{243}$. Thus, the interval pattern for Pythagorean tuning is big-big-small-big-big-big-small. Writing this in terms of tones and semitones, we get tone-tone-semitone-tone-tone-tone-semitone. This pattern holds for all scales in the Pythagorean tuning system.

We name the intervals based on the note number of the higher note. So the interval between the fifth and fourth note is called the fifth.

Let's take a look at the keys on a piano to apply the interval pattern to Example F.



Like we found before, the notes for the E scale are E, F♯, G♯, A, B, C♯, D♯, E.

Exercise 5

Find the intervals of the just intonation system.

Stop and Think

Is it possible to find a tone-semitone pattern for the just intonation system?

Problems with These Tuning Systems

After learning about Pythagorean tuning and just intonation, you may be wondering which one we use in today's music? The answer is neither one and we will discuss some problems in both of these tuning systems.

Pythagorean Tuning

One major problem of the Pythagorean tuning system has to do with transposition. If we started with any scale and kept transposing it up by a fifth over and over until we get back to the original tonic, we would not end up with the same notes of the original scale. In fact, the new tonic will not be an octave multiple of the original tonic. Another issue as we discussed is that the numbers in the fractions are quite large. If we transposed a Pythagorean scale over and over again, the calculations would become very tedious. Another issue arises in certain keys of Pythagorean scales where we get something called a wolf interval. Take a listen to this clip to see what this sounds like.

Stop and Think

Does the wolf interval sound good or bad to you?

Just Intonation

Not only does just intonation use smaller numbers in its fractions, but it sounds lovely and harmonious to listen to. So what could be the issue with it? Take a listen to the just scales in the key of C and in the key of D to discover the issue with just intonation.

As you just heard, just intonation makes a beautiful sound in certain keys but doesn't work in other keys. There also becomes an issue with the notes being played in certain just intonation keys. Given any tonic, we are able to easily calculate the rest of the notes with the fractions. But the problem is that we will end up with some frequencies that don't translate to any of our notes from A to G nor any of the sharpened or flattened versions of those notes. So the scale would be impossible to play on a normal piano.

Stop and Think

Do you think it's possible to change the tone of the keys on a piano? What about the strings on a guitar?

The Ideal Tuning System

Our ideal tuning system should be able to be played in every key and we also want it to sound good. As an extra bonus, it would be nice if the fractions contained small numbers. As we all know, nothing is perfect, even in the world of music, and so compromises must be made. People have decided over time that it's better to sacrifice some of the purity of the notes in order to have flexibility to play in any key. This way, we don't have to switch between tuning systems to play in a different key which makes it easier for musicians to play instruments and sing.

Equal Temperament

In the late 1500s, there was an idea to equally divide the 12 semitones in an octave among the intervals. This changed the value of a semitone and so a new unit for measuring intervals was created: **cents**. This new tuning system is called **equal temperament**. In equal temperament, the octave is measured as 1200 cents with each semitone measuring 100 cents and each tone measuring 200 cents. The same tone-tone-semitone-tone-tone-tone-semitone pattern from Pythagorean tuning is kept. The table below describes the intervals in equal temperament.

do	re	mi	fa	so	la	ti	do
-	200 cents	400 cents	500 cents	700 cents	900 cents	1100 cents	1200 cents

Equal temperament has many benefits. It's easy to remember the intervals, it works and sounds "good" in every key, and can be tuned to all common instruments like the guitar and piano. For these reasons, equal temperament is the system we use for tuning in most of the Western music we hear today. A drawback is that its notes don't sound as "pure" as they do in just intonation or Pythagorean tuning. Listen to this clip to hear the difference between just intonation and equal temperament. Do you prefer one over the other? Another observation is that all the keys sound the same and lack character. Have you ever thought 2 songs on the radio sounded similar? Or you may have even heard of some copyright disputes between artists claiming that their song was stolen by another artist. The nature of equal temperament is to blame for this. Since all keys have the same sound, a lot of songs will have a similar sound to each other. In this system, there are also only a finite combination of notes we can use to create a new song.

Final Thoughts

Over time, people have decided that there is music that sounds "good" and music that sounds "bad". Of course everyone has their own opinions on this and there is no set definition of "good" and "bad" in music. In fact, there are whole genres dedicated to having more unpleasant and atypical sounds. Although equal temperament is the most common today, there are still artists and musicians who use other tuning systems to achieve a certain sound. Everyone has a preference and some people even enjoy the sound of just intonation played in the "wrong" key. Music is all subjective and there is no right or wrong way of listening to it.